

# SPACE-FILLING CURVES OF SELF-SIMILAR SETS (II): FROM FINITE SKELETON TO SUBSTITUTION RULE

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**ABSTRACT:** This paper is one of the four papers in a series, which provide a systematic treatment for the space-filling curves (or optimal parameterizations) of self-similar sets.

It is folklore that the constructions of space-filling curves depend on certain ‘substitution rules’. For a given self-similar set, how to obtain such substitution rules is a long-standing open problem. In this paper, we obtain a satisfactory answer to this problem. We introduce the *finite skeleton property* for an iterated function system (IFS). From a finite skeleton, we construct a graph  $G_0$  and its ‘iteration’ graph  $G_1$ ; By studying the relation between  $G_0$  and  $G_1$ , we obtain define various substitution rules. We show that is a rule is consistency and primitive, then a space-filling curve of the self-similar set can be constructed accordingly.

In a sequential paper [41], we show that there always exists consistency and primitive rule, and hence finite skeleton property leads to optimal parameterizations. Furthermore, in [40], we show that a connected self-similar set with finite type condition always possesses finite skeletons.

Our results cover a large class of self-similar sets, and extend almost all the previous studies in this direction. All the proofs are constructive, and hence an algorithm of constructing space-filling curves is obtained automatically.

## 1. Introduction

The subject of space-filling curves has fascinated mathematicians for over a century, since Peano’s monumental work [34] in 1890. After that, many space-filling curves have been constructed, for example, Hilbert curve (1891), Sierpiński curve (1912) (it is generalized by Pólya), Heighway dragon curve (of a 2-reptile, 1967), Gosper curve (of a 7-reptile, 1973). See [21, 47, 36, 10, 16, 9]. A survey of the early results can be found in Sagan [43].

After that, more systematic methods have been found: the  $L$ -system method introduced by Lindenmayer [26] (1968), and the recurrent set method introduced by Dekking [11] (1982).

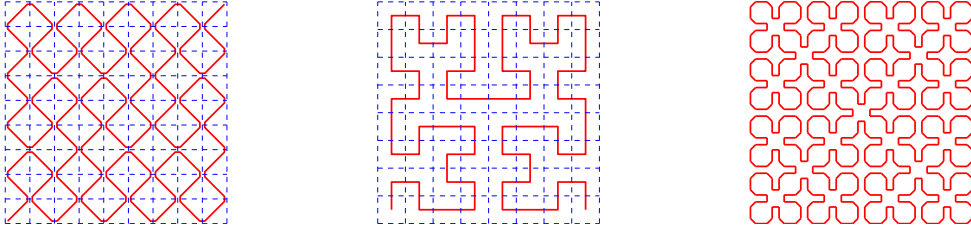


FIGURE 1. The constructions of Peano, Hilbert and Sierpiński.

In recent years, various interesting constructions of space-filling curves appear on the internet, for example, “[www.fractalcurves.com](http://www.fractalcurves.com)” ([53]) and “[teachout1.net/village/](http://teachout1.net/village/)” ([52]). Besides, space-filling curves of higher dimensional cubes have been studied by Milne [31] and Gilbert [19]. For applications of space-filling curves, see Bader [3] and the references therein.

The main purpose of this paper is to study the space-filling curves of self-similar sets. Following [39], the terminology space-filling curve is used in a very strong sense: it is a parametrization which is almost one-to-one, measure-preserving and  $1/s$ -Hölder continuous (See [39] for precise definition).

Let  $K \subset \mathbb{R}^d$  be a non-empty compact set. We call  $K$  a self-similar set, if it is a union of small copies of itself, precisely, there exist similitudes  $S_1, \dots, S_N : \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that

$$K = \bigcup_{j=1}^N S_j(K).$$

In fractal geometry, the family  $\{S_1, \dots, S_N\}$  is called an *iterated function system*, or IFS in short;  $K$  is called the *invariant set* of the IFS [22, 13]. We denote by  $\mathcal{H}^s$  the  $s$ -dimensional Hausdorff measure. A set  $E \subset \mathbb{R}^d$  is called an *s-set*, if  $0 < \mathcal{H}^s(E) < \infty$  for some  $s \geq 0$ .

The IFS  $\{S_1, \dots, S_N\}$  is said to satisfy the *open set condition (OSC)*, if there is an open set  $U$  such that  $\bigcup_{i=1}^N S_i(U) \subset U$  and the sets  $S_i(U)$  are disjoint. It is well-known that, if a self-similar set  $K$  satisfies the open set condition, then it is an  $s$ -set. (See [13].)

If an IFS satisfies the OSC condition, and  $\dim_H K$  equals the space dimension, then it has non-empty interior ([46]), and  $K$  is a self-similar tile. Especially, if the contraction ratios of  $S_i$  are all equal to  $r$ , then  $K$  is called a *reptile*.

In [39], we introduce a notion of linear GIFS to describe and handle space-filling curves of self-similar sets. A linear GIFS, roughly speaking, is a graph-directed iterated function system (GIFS in short), with a nice order structure. (See Section 2 for precise definition). It is shown that

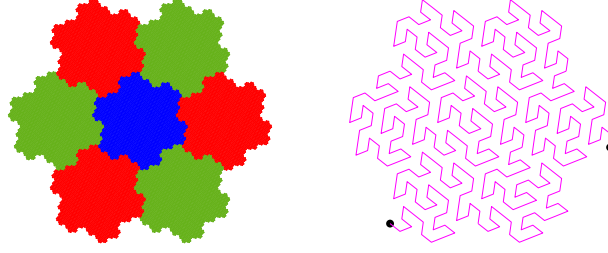


FIGURE 2. The Gosper island

**Theorem 1.1.** ([39]) *Let  $\{E_j\}_{j=1}^N$  be the invariant sets of a linear GIFS satisfying the open set condition and  $0 < \mathcal{H}^\delta(E_j) < \infty$  for  $1 \leq j \leq N$ , where  $\delta$  is the similarity dimension, then  $E_j$  admits optimal parametrizations for every  $j = 1, \dots, N$ .*

Thanks to the above theorem, to construct space-filling curves of a self-similar set, it is amount to seek linear GIFS structures of it, which is the main theme of the present paper.

**1.1. Skeleton of a self-similar set.** To explore GIFS structures of a self-similar set, we introduce the notion of skeleton of a self-similar set, which is crucial in this paper.

Let  $\{S_j\}_{j=1}^N$  be an IFS with invariant set by  $K$ . For any  $F \subset K$ , we define a graph  $H(F)$  with vertex set  $\{S_1, S_2, \dots, S_N\}$  as follows: there is an edge between two vertices  $S_i$  and  $S_j$  if and only if  $S_i(F) \cap S_j(F) \neq \emptyset$ . We call  $H(F)$  the *Hata graph* induced by  $F$ . Hata [20] proved that

**Proposition 1.1.** *a self-similar set  $K$  is connected if and only if the graph  $H(K)$  is connected.*

Motivated by this result, we propose the following definition.

**Definition 1.1.** Let  $A$  be a subset of  $K$ . We call  $A$  a *skeleton* of  $\mathcal{S}$  (or of  $K$ ), if the connecting graph  $H(A)$  is connected and  $A \subset \bigcup_{j=1}^N S_j(A)$ .

A self-similar set may have many skeletons. Figure 3 illustrates four skeletons of the Sierpiński carpet. Moran [32] and Kigami [23] have studied the ‘boundary’ or ‘vertices’ of a fractal. The skeleton is closely related to these conceptions but it is much different from them.

**1.2. Substitution rule.** From now on, we always assume that  $K$  is a self-similar set having the finite skeleton property and satisfying the open set condition. We denoted by  $A = \{a_1, \dots, a_m\}$  a finite skeleton.

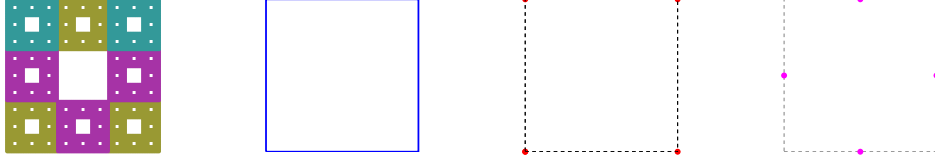


FIGURE 3. The carpet itself, the boundary of the unit square, four corner points of the unit square and the four middle points of the edges are all skeletons.

Let  $G_0 = (A, V_0)$  be the directed complete graph with vertex set  $A$ , that is,

$$V_0 = \{\overrightarrow{a_i a_j}; 1 \leq i, j \leq m\}.$$

We remark here the edges  $\overrightarrow{a_i a_j}$  should be thought as abstract edges. Next, we set

$$(1.1) \quad G_1 = \bigcup_{i=1}^N S_i(G_0)$$

to be the union of affine copies of the  $G_0$ . (See Section 6.)

Let  $V$  be a subset of  $V_0$ . For each  $e \in V$ , we associated with a path in  $G_1$  such that it has the same initial and terminate points as  $e$ . If in addition, all the edges in the paths are affine images of some elements in  $V$ , then we obtain a kind of substitution rule over  $V$ . We show that a edge-substitution rule induces a linear GIFS (Theorem 3.1). So, we turn to look for edge-substitution rules leading to optimal parameterizations of  $K$ .

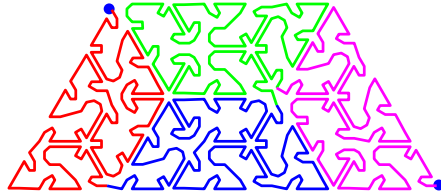


FIGURE 4. Space-filling curve of a wedge tile [52]. The detail is given in Example 4.1.

Construction of classical space-filling curves Three methods.

**1.3. Eulerian path method.** To find feasible substitution rules for all reasonable self-similar sets, we propose the *cut-and-glue* method. Fix the order of elements of  $A$ , and let  $\Lambda_0$  be a simple loop passing through  $a_1, \dots, a_m$  in turn. We choose  $V$  to be the edges  $\Lambda_0$



together with that of the reverse graph  $\Lambda_0^{-1}$ , i.e.,

$$(1.2) \quad V = \{\overrightarrow{a_i a_{i+1}}; i = 1, \dots, m\} \cup \{\overrightarrow{a_{i+1} a_i}; i = 1, \dots, m\},$$

where we identify  $a_{m+1}$  with  $a_1$  for simplicity; set

$$G = \bigcup_{i=1}^N S_j(V).$$

Secondly, choose a vector  $\beta = (\beta_1, \dots, \beta_N)$  over  $\{1, -1\}^N$  (which we call an orientation vector), and decompose the graph  $G$  into two parts as  $G = G(S, A, \beta) \cup G(S, A, -\beta)$ , where

$$G(S, A, \beta) = \bigcup_{j=1}^N S_j(\Lambda_0^{\beta_j})$$

is the the union of similar copies of  $\Lambda_0$  and  $\Lambda_0^{-1}$ . We show that the problem of choosing a substitution rule can be simplified to select a feasible unicursal path (a path passes every edge but passes only once) of  $G(S, A, \beta)$ . This can be thought as cutting the small loops  $S_j(\Lambda_0^{\beta_j})$  and then gluing them up.

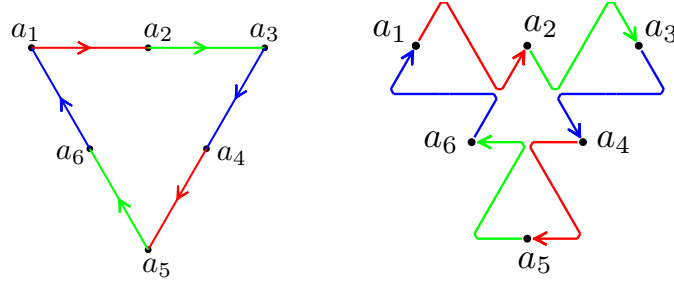


FIGURE 5. The four tile star. This example is taken from the web-site [52].

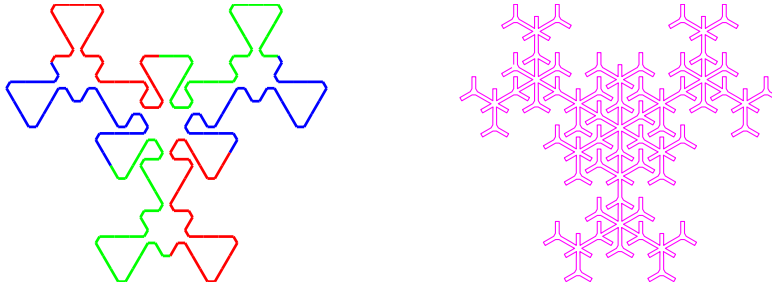


FIGURE 6.

Thirdly, we explore conditions on unicursal path  $P$  of  $G(S, A, \beta)$  to guarantee the induced substitution rule is good. (The terminologies in the following theorem are defined in Section 5.)

**Theorem 1.2.** *Let  $\{S_i\}_{i=1}^N$  be an IFS having the finite skeleton property and satisfying the open set condition. If there exist a finite skeleton  $A$ , an orientation vector  $\beta$  and a unicursal path  $P$  of  $G(S, A, \beta)$  such that a partition  $P = P_1 + \cdots + P_m$  is consistent, primitive and contains pure cells, then*

- (i) *The GIFS induced by  $P$  is a linear GIFS satisfying the OSC.*
- (ii)  *$K = \bigcup_{j=1}^m E_{v_j}$  and the union is disjoint in  $s$ -dimensional Hausdorff measure, where  $\{E_{v_j}\}_{j=1}^m$  are invariant sets of the induced GIFS, and  $s = \dim_H K$ .*
- (iii)  *$K$  admits optimal parameterizations.*

In some cases, choosing  $\beta = (1, \dots, 1)$ , we can find an Eulerian path satisfying the requirements of the above theorem; for example, the square, the ter-dragon, the four star tile, the Sierpiński carpet, etc. Figure 5 shows the graphs  $\Lambda_0$ ,  $G(S, A, \beta)$ , and a unicursal path together with its decomposition.

Fourthly, in a sequential paper [41], we show that the condition (i)-(iii) in the above theorem are superfluous if we iterated the IFS enough times, and hence the above theorem can be strengthened to

**Theorem 1.3.** *If a self-similar set  $K$  has the finite skeleton property and satisfies the open set condition, then it admits optimal parametrizations.*

Self-similar sets satisfying the *finite type condition* are the most important class of fractals ([38, 28, 4, 6]). It is shown that

**Lemma 1.2.** ([40]) *A self-similar set has the finite skeleton property if it is connected and fulfills the finite type condition.*

As a consequence of the above two theorems, we have

**Theorem 1.4.** *If a connected self-similar set satisfies the finite type condition and the open set condition, then it admits optimal parameterizations.*

The paper is organized as follows. In Section 2, we introduce the notions of linear GIFS. In Section 3, we introduce the general idea of constructing linear GIFS using skeletons. In Section 4, we revisit the classical constructions of space-filling curves from our new

point of views. In Section 5 we discuss the Eulerian-path method and Theorem 1.2 is proved there.

## 2. Linear GIFS

**2.1. GIFS.** Let  $G = (\mathcal{A}, \Gamma)$  be a directed graph with vertex set  $\mathcal{A}$  and edge set  $\Gamma$ . Let

$$\mathcal{G} = \left( g_\gamma : \mathbb{R}^d \rightarrow \mathbb{R}^d \right)_{\gamma \in \Gamma}$$

be a family of similitudes. We call the triple  $(\mathcal{A}, \Gamma, \mathcal{G})$ , or simply  $\mathcal{G}$ , a *graph-directed iterated function system* (GIFS).

We call  $(\mathcal{A}, \Gamma)$  the *base graph* of the GIFS. In what follows, we shall call  $\mathcal{A}$  a *state set* instead of a vertex set, to avoid confusion with graphs in Euclidean space we introduce later. Very often but not always, we set  $\mathcal{A}$  to be  $\{1, \dots, N\}$ .

Let  $\Gamma_{ij}$  be the set of edges from state  $i$  to  $j$ . It is well known that there exist unique non-empty compact sets  $\{E_i\}_{i=1}^N$  satisfying

$$(2.1) \quad E_i = \bigcup_{j=1}^N \bigcup_{\gamma \in \Gamma_{ij}} g_\gamma(E_j), \quad 1 \leq i \leq N.$$

We call  $\{E_j\}_{j=1}^N$  the *invariant sets* of the GIFS [29].

**Remark 2.1.** The set equations (2.1) give all the information of a GIFS, and hence provide an alternative way to define a GIFS. We shall call (2.1) the *set equation form* of a GIFS.

**2.2. Symbolic space related to a graph  $G$ .** A sequence of edges in  $G$ , denoted by  $\omega = \omega_1 \omega_2 \dots \omega_n$ , is called a *path*, if the terminate state of  $\omega_i$  coincides with the initial state of  $\omega_{i+1}$  for  $1 \leq i \leq n-1$ . We will use the following notations to specify the sets of finite or infinite paths on  $G = (\mathcal{A}, \Gamma)$ . For  $i \in \mathcal{A}$ , let

$$\Gamma_i^k, \Gamma_i^* \text{ and } \Gamma_i^\infty$$

be the set of all paths with length  $k$ , the set of all paths with finite length, and the set of all infinite paths, emanating from the state  $i$ , respectively. Note that  $\Gamma_i^* = \bigcup_{k \geq 1} \Gamma_i^k$ .

For an infinite path  $\omega = (\omega_n)_{n=1}^\infty \in \Gamma_i^\infty$ , we set  $\omega|_k = \omega_1 \omega_2 \dots \omega_k$  and call

$$[\omega_1 \dots \omega_n] := \{\gamma \in \Gamma_i^\infty; \gamma|_n = \omega_1 \dots \omega_n\}$$

the *cylinder* associated with  $\omega_1 \dots \omega_n$ .

For a path  $\gamma = \gamma_1 \dots \gamma_n$ , we denote

$$E_\gamma := g_{\gamma_1} \circ \dots \circ g_{\gamma_n}(E_{t(\gamma)}),$$

where  $t(\gamma)$  denotes the terminate state of the path  $\gamma$  (and  $\gamma_n$ ). Iterating (2.1)  $k$ -times, we obtain

$$(2.2) \quad E_i = \bigcup_{\gamma \in \Gamma_i^k} E_\gamma.$$

We define a projection  $\pi : (\Gamma_1^\infty, \dots, \Gamma_N^\infty) \rightarrow (\mathbb{R}^d, \dots, \mathbb{R}^d)$ , where  $\pi_i : \Gamma_i^\infty \rightarrow \mathbb{R}^d$  is defined by

$$(2.3) \quad \{\pi_i(\omega)\} := \bigcap_{n \geq 1} E_{\omega|_n}.$$

For  $x \in E_i$ , we call  $\omega$  a coding of  $x$  if  $\pi_i(\omega) = x$ . It is folklore that  $\pi_i(\Gamma_i^\infty) = E_i$ .

**2.3. Order GIFS and linear GIFS.** Let  $(\mathcal{A}, \Gamma, \mathcal{G})$  be a GIFS. To study the ‘advanced’ connectivity property of the invariant sets, we equip a partial order on the edge set  $\Gamma$  enlightened by set equation (2.2). Let  $\Gamma_i = \Gamma_i^1$  be the set of edges emanating from the state  $i$ .

**Definition 2.1.** We call the quadruple  $(\mathcal{A}, \Gamma, \mathcal{G}, \prec)$  an *ordered GIFS*, if  $\prec$  is a partial order on  $\Gamma$  such that

- (i)  $\prec$  is a linear order when restricted on  $\Gamma_j$  for every  $j \in \mathcal{A}$ ;
- (ii) elements in  $\Gamma_i$  and  $\Gamma_j$  are not comparable if  $i \neq j$ .

The order  $\prec$  induces a *dictionary order* on each  $\Gamma_i^k$ , namely,  $\gamma_1 \gamma_2 \dots \gamma_k \prec \omega_1 \omega_2 \dots \omega_k$  if and only if  $\gamma_1 \dots \gamma_{\ell-1} = \omega_1 \dots \omega_{\ell-1}$  and  $\gamma_\ell \prec \omega_\ell$  for some  $1 \leq \ell \leq k$ . Observe that  $(\Gamma_i^k, \prec)$  is a linear order. Now we can define the linear GIFS.

**Definition 2.2.** Let  $(\mathcal{A}, \Gamma, \mathcal{G}, \prec)$  be an ordered GIFS with invariant sets  $\{E_i\}_{i=1}^N$ . It is termed a *linear GIFS*, if for all  $i \in \mathcal{A}$  and  $k \geq 1$ ,

$$E_\gamma \cap E_\omega \neq \emptyset$$

provided  $\gamma, \omega$  are adjacent paths in  $\Gamma_i^k$ .

**2.4. Chain condition.** Let  $(\mathcal{A}, \Gamma, \mathcal{G}, \prec)$  be an ordered GIFS. Denote the invariant sets by  $\{E_i\}_{i \in \mathcal{A}}$ . For an edge  $\omega \in \Gamma$ , recall that  $g_\omega$  is the associated similitude and  $t(\omega)$  is the terminate state.

For  $i \in \mathcal{A}$ , a path  $\omega \in \Gamma_i^\infty$  is called the *lowest* path, if  $\omega|_n$  is the lowest path in  $\Gamma_i^n$  for all  $n$ ; in this case, we call  $a = \pi_i(\omega)$  the *head* of  $E_i$ . Similarly, we define the highest path  $\omega'$  of  $\Gamma_i^\infty$ , and we call  $b = \pi_i(\omega)'$  the *tail* of  $E_i$ .

**Definition 2.3.** An ordered GIFS is said to satisfy the *chain condition*, if for any  $i \in \mathcal{A}$ , and any two adjacent edges  $\omega, \gamma \in \Gamma_i$  with  $\omega \prec \gamma$ ,

$$g_\omega(\text{tail of } (E_{t(\omega)})) = g_\gamma(\text{head of } E_{t(\gamma)}).$$

**Theorem 2.1.** *An ordered GIFS is a linear GIFS if and only if it satisfies the chain condition.*

### 3. From finite skeleton to substitution rule

In this section, we discuss how to find substitution rules when a self-similar set with a finite skeleton is given.

Let  $\mathcal{S} = \{S_j\}_{j=1}^N$  with invariant set  $K$ . From now on, we always assume that  $\mathcal{S}$  possesses finite skeleton property and satisfies the OSC. Let  $A = \{a_1, a_2, \dots, a_m\}$  be a skeleton, and we fix the order of  $A$ .

Let us denote

$$V_0 = \{\overrightarrow{a_i a_j}; 1 \leq i, j \leq m\}$$

where  $\overrightarrow{a_i a_j}$  denotes the edge from  $a_i$  to  $a_j$ . Then  $G_0 = (A, V_0)$  is a directed complete graph. We remark here the edges  $\overrightarrow{a_i a_j}$  should be thought as abstract edges.

To continue our construction, we need to define the *affine copy* of a directed graph.

**Definition 3.1.** Let  $G = (\mathcal{A}, \Gamma)$  be a directed graph such that  $\mathcal{A} \subset \mathbb{R}^d$ . Let  $S : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a similitude. We define a directed graph  $G_S = (\mathcal{A}_S, \Gamma_S)$  as follows:

- (i) The vertex set is  $\mathcal{A}_S = S(\mathcal{A}) = \{S(x); x \in \mathcal{A}\}$ ;
- (ii) There is an edge in  $\Gamma_S$  from  $S(x)$  to  $S(y)$ , if and only if there is an edge  $e \in \Gamma$  from vertex  $x$  to  $y$ . Moreover, we denote this edge by  $(e, S)$ .

For simplicity, we shall denote  $G_S, \mathcal{A}_S, \Gamma_S$ , and  $(e, S)$  by  $S(G), S(\mathcal{A}), S(\Gamma)$  and  $S(e)$ , respectively. We define

$$G_1 = \bigcup_{i=1}^N S_i(G_0)$$

to be the union of affine copies of  $G_0$ . The edge set of  $G_1$  is  $\{S_i(v); 1 \leq i \leq N, v \in V_0\}$ .

We remark that if  $(\mathcal{A}_1, \Gamma_1)$  and  $(\mathcal{A}_2, \Gamma_2)$  are two graphs without common edges, their union is defined to be the graph  $(\mathcal{A}_1 \cup \mathcal{A}_2, \Gamma_1 \cup \Gamma_2)$ . Even if  $S_{j'}(e_k)$  coincides with  $S_j(e_{k'})$  as oriented line segment, they should be regarded as different edges since  $(e_k, S_{j'}) \neq (e_{k'}, S_j)$ .

**Remark 3.1.** Inductively, we can define

$$G_n = \bigcup_{i=1}^N S_i(G_{n-1}), \quad n \geq 1.$$

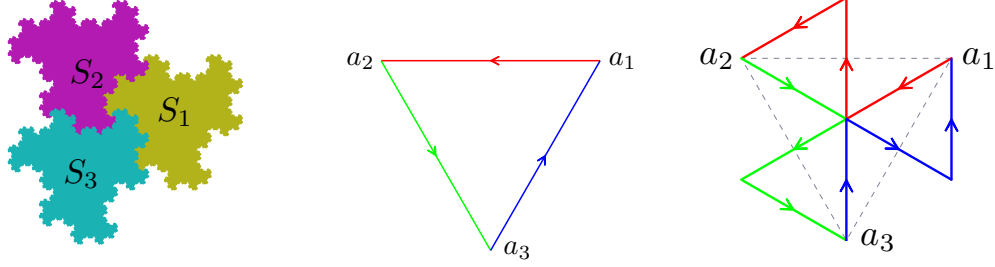


FIGURE 7. Union of affine copies of a graph.

**3.1. Substitution rules.** Let  $V$  be a subset of  $V_0$ . Let  $\tau$  be a mapping from  $V$  to paths of  $G_1$ . We shall denote  $\tau(u)$  by  $P_u$  to emphasize that  $\tau(u)$  is a path, that is,

$$(3.1) \quad \tau : u \mapsto P_u, \quad u \in V.$$

**Definition 3.2.** The above mapping  $\tau$  is called a *substitution rule*, if

- (i)  $P_u$  has the same initial and terminate points as  $u$ ;
- (ii) all the edges of  $P_u$  have the form  $S_i(v)$  for some  $1 \leq i \leq N$  and  $v \in V$ , or equivalently,  $P_u$  is a path in the graph

$$G = \bigcup_{i=1}^N S_i(V).$$

A substitution rule  $\tau$  can be thought as replacing each big edge  $u$  by a path  $P_u$  consisting of small edges, while the initial point and the terminate point remain unchanged.

**Example 3.1. Terdragon (1).** Terdragon can be generated by the IFS

$$\{S_1(z) = \lambda z + 1, S_2(z) = \lambda z + \omega, S_3(z) = \lambda z + \omega^2\},$$

where  $\lambda = \exp(\pi i/6)/\sqrt{3}$  and  $\omega = \exp(2\pi i/3)$ .

Let  $A = \{a_1, a_2, a_3\} = \{-\omega^2/\lambda, -1/\lambda, -\omega/\lambda\}$ , which are the fixed point of  $S_1, S_2$  and  $S_3$ , respectively. Since  $S_1(a_2) = S_2(a_3) = S_3(a_1) = 0$ , one deduces that  $A$  is a skeleton of the terdragon.

Let  $V$  be the graph indicated by Figure 7(middle), then

$$G = S_1(V) \cup S_2(V) \cup S_3(V)$$

is the graph in Figure 7(right). Let us denote  $v_1 = \overrightarrow{a_1 a_2}$ ,  $v_2 = \overrightarrow{a_2 a_3}$ ,  $v_3 = \overrightarrow{a_3 a_1}$ , then

$$(3.2) \quad \begin{cases} v_1 \mapsto P_{v_1} = S_1(v_1)S_2(v_3)S_2(v_1), \\ v_2 \mapsto P_{v_2} = S_2(v_2)S_3(v_1)S_3(v_2), \\ v_3 \mapsto P_{v_3} = S_3(v_3)S_1(v_2)S_1(v_3), \end{cases}$$

is a substitution rule.

**Remark 3.2.** Now we define the iteration of  $\tau$ . For  $I \in \{1, 2, \dots, N\}^*$  and  $u \in V$ , we set

$$\tau(S_I(u)) = S_I(\tau(u)) \text{ and } \tau(w_1 w_2 \dots w_k) = \tau(w_1) \tau(w_2) \dots \tau(w_k)$$

where  $w_j = S_{I_j}(u_j)$ . Then  $\tau^n(u)$  is a path in  $G_n$  which can be regarded as an oriented broken line.

**3.2. Induced GIFS.** Now we construct an ordered GIFS according to a substitution rule  $\tau$ . The path  $P_u$  can be written as

$$P_u = S_{u,1}(v_{u,1}) \dots S_{u,\ell_u}(v_{u,\ell_u}),$$

where  $S_{u,j} \in \mathcal{S}$  and  $v_{u,j} \in V$  for  $j = 1, \dots, \ell_u$ . Replacing  $u$  by  $E_u$  in the substitution rule (3.1), we obtain the following ordered GIFS:

$$(3.3) \quad E_u = S_{u,1}(E_{v_{u,1}}) + \dots + S_{u,\ell_u}(E_{v_{u,\ell_u}}), \quad u \in V.$$

Now we give a description of the base graph of the above GIFS in detail. Clearly, the state set is  $V$ . From now on, we call an edge between two states in  $V$  a *bridge*, to distinguish them from the edges in  $G_0$  or  $G_1$ . Since the  $k$ -th term on the right hand side of (3.3) is  $S_{u,k}(v_{u,k})$ , we denote the  $k$ -th bridge emanating from  $u$  by the quadruple

$$(3.4) \quad (u, k, S_{u,k}, v_{u,k}),$$

whose components are the initial state, the order, the associated contraction mapping and terminate state of the bridge, respectively. This defines the bridge set, and we denote it by  $\mathcal{E}$ . We shall denote the GIFS (3.3) by

$$(3.5) \quad (V, \mathcal{E}, \mathcal{G}, \prec),$$

and call it the *GIFS induced by the substitution rule  $\tau$* . Indeed, if we denote the bridge in (3.4) by  $\omega$ , then

$$(3.6) \quad g_\omega = S_{u,k}, \quad t(\omega) = v_{u,k}$$

where  $t(\omega)$  denotes the terminate state of the bridge  $\omega$ .

Next, we show that

**Theorem 3.1.** *The induced GIFS  $(V, \mathcal{E}, \mathcal{G}, \prec)$  in (3.5) is a linear GIFS.*

*Proof.* Let  $u \in V$ . We denote by  $a_u$  and  $b_u$  the initial and terminate point of  $u$  as an edge in the graph  $G_0$ . Let  $\omega = (u, 1, S, v)$  be the first bridge emanating from  $u$  in the GIFS (3.5). Then

$$(3.7) \quad a_u = S(a_v),$$

since the initial point of  $u$  coincides with that of the first edge in the path  $P_u$ .

Recall that the head of  $E_u$  is point in  $E_u$  possessing the lowest coding. (See Section 3.) We claim that

*Claim:* For any  $u \in V$ ,  $a_u$  and  $b_u$  are head and tail of  $E_u$ , respectively.

Let  $\omega = (\omega_k)_{k=1}^\infty$  be the lowest path starting from the state  $u$ . Then  $\omega_1$  has the form  $\omega_1 = (u, 1, S_{u,1}, v_{u,1})$ . By (3.7),

$$a_u = S_{u,1}(a_{v_{u,1}}) = g_{\omega_1}(a_{t(\omega_1)}),$$

where  $t(\omega_1)$  denotes the terminate state of  $\omega_1$ .

Since  $(\omega_k)_{k=2}^\infty$  is the lowest path starting from  $v_{u,1}$ , by the same reason as above,

$$a_{t(\omega_1)} = g_{\omega_2}(a_{t(\omega_2)}).$$

The above two formulas imply that  $a_u = g_{\omega_1} \circ g_{\omega_2}(a_{t(\omega_2)})$ . Repeating this argument, we obtain  $a_u = g_{\omega_1} \circ g_{\omega_2} \cdots \circ g_{\omega_n}(a_{t(\omega_n)})$  for all  $n \geq 1$ . It follows that

$$\{a_u\} = \bigcap_{n=1}^{\infty} g_{\omega|_n}(A)$$

where  $A = \{a_1, \dots, a_m\}$  is the skeleton. That is,  $\omega$  is a coding of  $a_u$ , which proves that  $a_u$  is the head of  $E_u$ . The second assertion can be proved by the same argument. The claim is proved.

For  $u \in V$ , let  $\omega$  and  $\gamma$  be two adjacent bridges in  $\mathcal{E}_u$  and  $\omega \prec \gamma$ . Then there exists  $k \in \{1, \dots, \ell_u\}$  such that

$$\omega = (u, k, S_{u,k}, v_{u,k}) \text{ and } \gamma = (u, k+1, S_{u,k+1}, v_{u,k+1}).$$

Since  $S_{u,k}(v_{u,k})$  and  $S_{u,k+1}(v_{u,k+1})$  are adjacent edges in the path  $P_u$ , we have

$$S_{u,k}(b_{v_{u,k}}) = S_{u,k+1}(a_{v_{u,k+1}}),$$

that is,  $S_{u,k}(\text{tail of } E_{v_{u,k}}) = S_{u,k+1}(\text{head of } E_{v_{u,k+1}})$ , so by (3.6),

$$g_\omega(\text{tail of } E_{t(\omega)}) = g_\gamma(\text{head of } E_{t(\gamma)}).$$

Thus the induced GIFS satisfies the chain condition, and it is linear.  $\square$



Now, we have to answer the following two questions, which are the main concern of the rest sections of the paper.

- (i) For which  $V$  and substitution rule  $\tau$  over  $V$ , the induced GIFS satisfies the OSC?
- (ii) When the optimal parameterizations of the invariant sets of the induced GIFS lead to an optimal parametrization of the original self-similar set  $K$ ?

#### 4. Classical space-filling curves

There are three methods of constructing classical space-filling curves: the path-on-lattice method, the traversing paths method and the unicursal-path method.

**4.1. Path-on-lattice method.** This method has been studied in detail in [39] and [?]. For a path-on-lattice IFS with terminate point  $d$ , we choose the skeleton  $A = \{a_1, a_2\} = \{0, d\}$ , and choose  $V = \{\overrightarrow{a_1 a_2}, \overrightarrow{a_2 a_1}\}$ . The fine substitution rule is given in a obvious way. (See Theorem 4.1.)

**4.2. Traversing-paths method.** Recall that  $G_1$  is the union of affine copies of the complete graph  $G_0$ . We say a path  $P$  on  $G_1$  is a *traversing path*, if for each  $i \in \{1, \dots, N\}$ , there exists but only one  $v \in V_0$  such that  $S_i(v)$  is an edge of  $P$ .

**Theorem 4.1.** *Let  $\{u \mapsto P_u; u \in V\}$  be a fine substitution rule. If all the paths  $P_u$  are traversing paths, then  $E_j = K$  for all  $j$  and the GIFS satisfies the open set condition. Consequently,  $K$  admits an optimal parametrization.*

*Proof.* Let us denote the induced GIFS by  $(V, \mathcal{E}, \mathcal{G}, \prec)$ . Fix a  $u \in V$ . Then the assumption implies that

$$\bigcup_{\omega \in \mathcal{E}_u^n} g_\omega(A) = \bigcup_{I \in \{1, \dots, N\}^n} S_I(A).$$

Taking the limit in Hausdorff metric at both sides, we obtain  $E_u = K$ . Moreover, if we forget the order structure, then the GIFS degenerates to the original IFS, so the OSC holds.  $\square$

**Example 4.1. The Wedge Tile.** The Wedge tile is a self-similar set, where the maps are indicated by Figure 8. Choose  $V = \{\overrightarrow{a_2 a_4}, \overrightarrow{a_4 a_2}, \overrightarrow{a_4 a_4}\}$ . Then

$$\tau : \begin{cases} \overrightarrow{a_4 a_2} \mapsto S_1(\overrightarrow{a_2 a_4}) S_2(\overrightarrow{a_2 a_4}) S_3(\overrightarrow{a_4 a_4}) S_4(\overrightarrow{a_4 a_2}), \\ \overrightarrow{a_2 a_4} \mapsto S_4(\overrightarrow{a_2 a_4}) S_3(\overrightarrow{a_4 a_4}) S_2(\overrightarrow{a_4 a_2}) S_1(\overrightarrow{a_4 a_2}), \\ \overrightarrow{a_4 a_4} \mapsto S_3(\overrightarrow{a_2 a_4}) S_4(\overrightarrow{a_4 a_4}) S_2(\overrightarrow{a_4 a_2}) S_1(\overrightarrow{a_4 a_2}). \end{cases}$$

is a fine substitution rule. Figure 9 shows the paths for  $\overrightarrow{a_4 a_2}$  and  $\overrightarrow{a_4 a_4}$ , the path for  $\overrightarrow{a_2 a_4}$  is the reverse path of  $\overrightarrow{a_4 a_2}$ . It is easy to verify that this substitution rule satisfies the

condition of Theorem 4.1. A visualization of the space-filling curve corresponding this linear GIFS is shown in Figure 4.

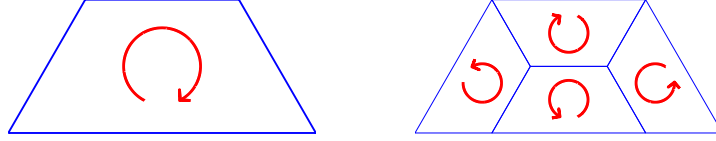


FIGURE 8.

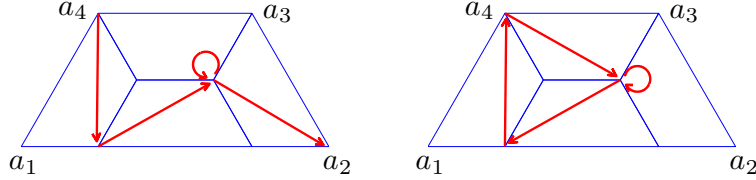


FIGURE 9.

**Example 4.2. The Hexaflake.** The Hexaflake is a fractal constructed by iteratively exchanging each hexagon by a flake of seven hexagons.(Figure 11 (left)). We choose  $v = \{\overrightarrow{a_4a_5}, \overrightarrow{a_4a_6}, \overrightarrow{a_4a_1} \dots\}$ . Figure 10 shows the paths for  $\overrightarrow{a_4a_5}$ ,  $\overrightarrow{a_4a_6}$  and  $\overrightarrow{a_4a_1}$ . The fine substitution rule can also be obtained from the figure. The substitution rule satisfies the condition of Theorem 4.1, and hence the Hexaflake has a linear structure. Figure 11 (middle) gives the visualization of the fractal-filling curve corresponding this linear GIFS.

As we have introduced that the cut-and-glue method is a almighty way to solve the parameterized problem of similarity sets. Figure 11 (right) gives us a visualization of fractal-filling curve using the unicursal-path method.

**Remark 4.1. Space-filling curves of polygonal reptiles.** In the web-site [52], there are space-filling curves of some polygonal reptiles, including the sphinx tile, the chair tile *etc.* These space-filling curves can be easily obtained from some edge-substitution rules. The regularity of the rules are guaranteed by Theorem 4.1.

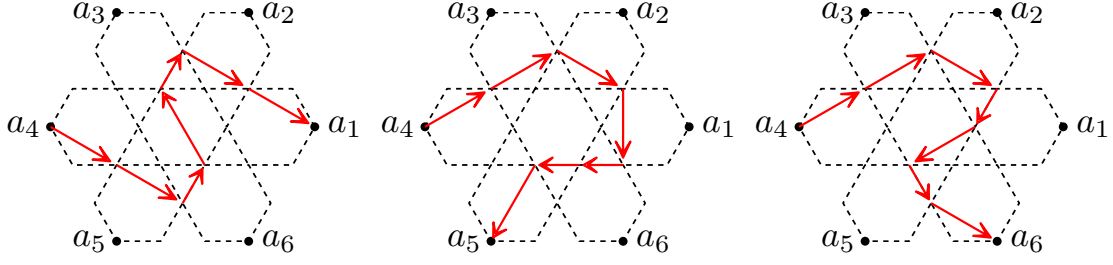


FIGURE 10.

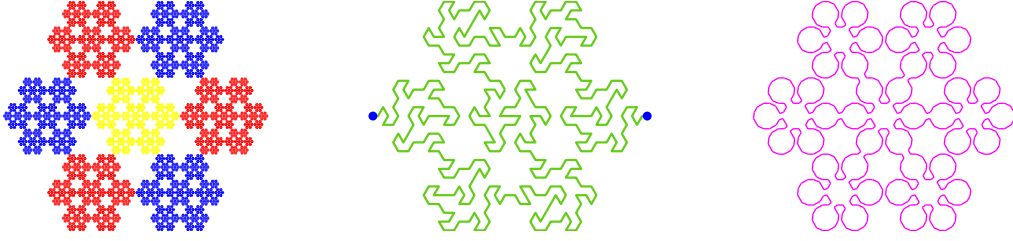


FIGURE 11.

**4.3. Eulerian path method: the simple version.** Recall that  $A = \{a_1, \dots, a_m\}$  is a finite skeleton. Let  $\Lambda_0$  be a simple loop passing through the vertices of the skeleton  $A$  in turn, that is,  $\Lambda_0 = v_1 + v_2 + \dots + v_m$ , where  $v_j = \overrightarrow{a_j a_{j+1}}$  are edges from  $a_j$  to  $a_{j+1}$  (here we shall identify  $a_{m+1}$  with  $a_1$  for simplicity). We choose

$$V = \{v_1, \dots, v_m\},$$

and call

$$G = \{S_i(v_j); 1 \leq i \leq N, 1 \leq j \leq m\}$$

the *simple induced graph*. We shall use this  $V$  to develop a fine substitution rule.

We assert that  $G$  admits a *unicursal path*, that is, a path visits all edges of  $G$  and visits them only once. Indeed,  $G$  can be regarded as the union of  $N$  loops  $S_i(\Lambda_0)$ . We shall call  $S_i(\Lambda_0)$  the  $S_i$ -cell. Starting from any cell, say,  $S_1(\Lambda_0)$ , we can add the other cells one by one in a way such that the union of the chosen cells are always connected. Suppose the  $S_j$ -cell and the  $S_1$ -cell intersect at a point  $x$ . We cut both loops at  $x$ , and then glue them to make a big loop. Repeating this procedure, we exhaust all the cells and obtain one big loop. This loop is a unicursal loop of  $G$ .

We say two paths in a graph are disjoint if they have no common edge. Let  $P_1, \dots, P_m$  be paths in  $G$ . If  $P = P_1 + P_2 + \dots + P_m$ , and  $P_j$  are pairwise disjoint paths, then we call  $P_1 + \dots + P_m$  a *partition* of  $P$ .

**Definition 4.1.** Let  $P$  be a unicursal path of  $G$ . We say  $P = P_1 + P_2 + \dots + P_m$  is a *consistent partition*, if  $P_i$  is a path from  $a_i$  to  $a_{i+1}$  for each  $i \in \{1, 2, \dots, m\}$ .

Let us assume that there exists a unicursal path  $P$  of the induced graph possessing a consistent partition  $P = P_1 + \dots + P_m$ . By setting  $P_{v_j} = P_j$  in the edge-substitution method, we obtain a fine substitution rule  $\tau$ , that is,

$$\tau : v_j \mapsto P_j, \quad j = 1, \dots, m.$$

Denote the length of  $P_j$  by  $\ell_j$ . Then the path  $P_j$  ( $j = 1, \dots, m$ ) has the form

$$(4.1) \quad P_j = S_{j,1}(v_{j,1})S_{j,2}(v_{j,2}) \cdots S_{j,\ell_j}(v_{j,\ell_j}),$$

where  $S_{j,k} \in \{S_1, \dots, S_N\}$ , and  $v_{j,k} \in V$  for  $1 \leq k \leq \ell_j$ .

The coarse substitution is a simplified version of  $\tau$  by concentrating on the ‘types’ of elements of  $\tau(v)$ ,  $v \in V$ . Precisely,

$$\tau^* : v_j \longrightarrow v_{j,1} \dots v_{j,\ell_j}, \quad j \in \{1, \dots, m\}.$$

Then  $\tau^*$  is a substitution over the alphabet  $V$ . We say  $\tau^*$  is *primitive*, if there exists a number  $n \geq 1$  such that for any  $u, v \in V$ ,  $u$  appears in  $(\tau^*)^n(v)$ . (See [37] for the general theory on substitution.)

By Theorem 3.1,  $\tau$  defines an induced GIFS and this GIFS is linear. Denote the invariant of the GIFS by  $\{E_{v_i}\}_{i=1}^m$ . We shall show in the next section that:

(i)  $K = \bigcup_{i=1}^m E_{v_i}$ , and the above union is disjoint in Hausdorff measure if  $\tau$  contains *pure cells*, that is, there exist  $v \in V$ ,  $n \geq 1$ , and  $I \in \{1, 2, \dots, N\}^*$  such that the edges of  $S_I(V)$  all belong to  $\tau^n(v)$ ; in this case we call  $S_I(V)$  a *pure cell*. For example, the small triangles in Figure 12 with the same color are pure cells.

(ii) If the coarse substitution  $\tau^*$  is primitive, then the GIFS satisfies the OSC.

Hence,  $K$  admits optimal parameterizations if  $\tau^*$  is primitive and  $\tau$  contains pure cells.

**Example 4.3. The Terdragon (2).** An consistent Eulerian path and a substitution rule  $\tau$  is given by (3.2). The corresponding coarse substitution is

$$\tau^* \begin{cases} v_1 \mapsto v_1 v_3 v_1; \\ v_2 \mapsto v_2 v_1 v_2; \\ v_3 \mapsto v_3 v_2 v_3. \end{cases}$$

It is easy to see that  $\tau^*$  is primitive. Figure 12 (the fifth one) depicts the third approximation and indicates that  $\tau$  contains pure cells.

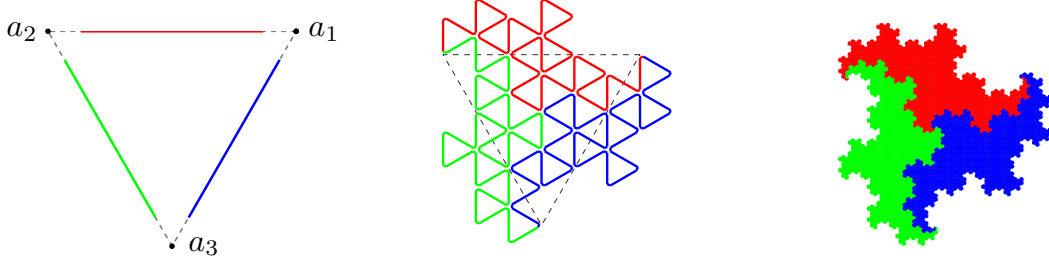


FIGURE 12. Terdragon

**Example 4.4. The Sierpiński carpet.** The functions in the IFS are all chosen to be homotheties. we choose the middle points of the four edges as the skeleton, then we obtain the parametrization indicated by Figure 13.

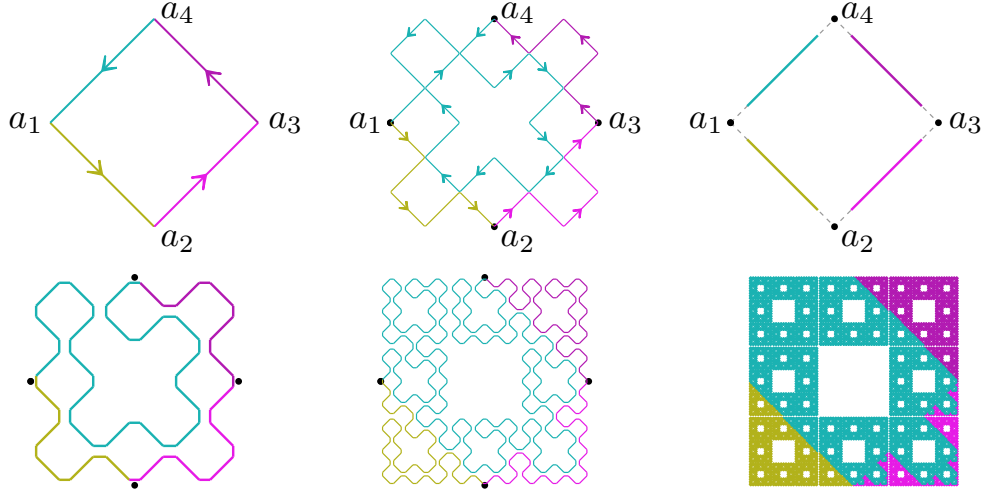


FIGURE 13. A fractal-filling curve of the Sierpiński carpet.

The four star tile can also be done by the cut-and-glue method, see Figure 5 and 6 in Section 1, and Example 4.5. However, the above construction will not always lead to an optimal parametrization.

**Example 4.5. The four star tile.** The four star tile is generated by the IFS

$$S_j(x) = -\frac{x}{2} + d_j, \text{ where } (d_1, d_2, d_3, d_4) = (0, e^{\pi i/6}, e^{5\pi i/6}, -i).$$

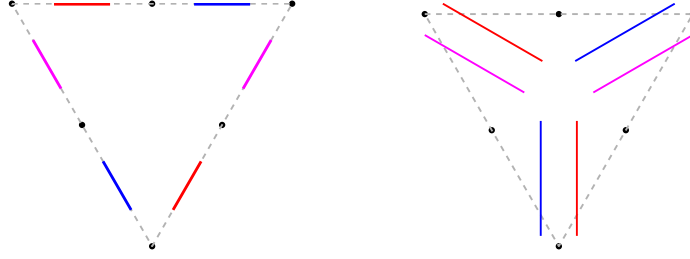


FIGURE 14. Initial patterns for space-filling curves of four star tile.

The linear GIFS is obtained by the cut-and-glue method (see the discussion of Section 7), and it is depicted in Figure 5. Figure 6 shows two visualizations of the space-filling curve, their initial patterns are illustrated by Figure 14.

### 5. Unicursal-path method: the general case

In this section, we discuss the unicursal-path method, and we shall show that this is an universal method.

Recall that  $A = \{a_1, \dots, a_m\}$  is an ordered finite skeleton. Denote

$$v_j = \overrightarrow{a_j a_{j+1}}, \quad v_j^{-1} = \overrightarrow{a_{j+1} a_j}, \quad j = 1, \dots, m.$$

(Here we shall identify  $a_{m+1}$  with  $a_1$  for simplicity), and we stipulate  $(v_j^{-1})^{-1} = v_j$ . Then  $\Lambda_0 = v_1 + v_2 + \dots + v_m$  is a loop. As we have mentioned in the introduction, choose

$$V = \{v_1, \dots, v_m\} \cup \{v_1^{-1}, \dots, v_m^{-1}\},$$

and define  $G = \bigcup_{i=1}^N S_i(V)$  to be the *induced graph*. We shall use this  $V$  and  $G$  to develop a substitution rule.

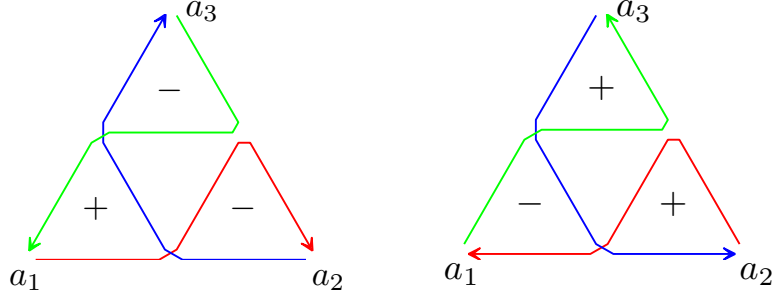
**5.1. Orientation.** Denote  $\Lambda_0^{-1} = v_m^{-1} + \dots + v_2^{-1} + v_1^{-1}$ . Let  $\beta = (\beta_1, \beta_2, \dots, \beta_N)$  be a vector in  $\{1, -1\}^N$ , and we call it an *orientation vector*. For any  $1 \leq j \leq N$ ,  $S_j(\Lambda_0^{\beta_j})$  is the affine copy of  $\Lambda_0^{\beta_j}$ . Denote

$$(5.1) \quad G(\mathcal{S}, A, \beta) = \bigcup_{j=1}^N S_j(\Lambda_0^{\beta_j}).$$

Then  $G$  can be decomposed into two graphs (see Figure 15)

$$(5.2) \quad G = G(\mathcal{S}, A, \beta) \cup G(\mathcal{S}, A, -\beta).$$

Hence, by symmetry, we need only to handle the graph  $G(\mathcal{S}, A, \beta)$ .

FIGURE 15.  $G(\mathcal{S}, A, \beta)$  and  $G(\mathcal{S}, A, -\beta)$ .

We assert that  $G$  admits a *unicursal path*, that is, a path visits all edges of  $G$  and visits them only once. Indeed,  $G$  can be regarded as the union of  $N$  loops  $S_i(\Lambda_0)$ . We shall call  $S_i(\Lambda_0)$  the  $S_i$ -cell. Starting from any cell, say,  $S_1(\Lambda_0)$ , we can add the other cells one by one in a way such that the union of the chosen cells are always connected. Suppose the  $S_j$ -cell and the  $S_1$ -cell intersect at a point  $x$ . We cut both loops at  $x$ , and then glue them to make a bigger loop. Repeating this procedure, we exhaust all the cells and obtain one big loop. This biggest loop is a unicursal loop of  $G$ .

We say two paths in a graph are disjoint if they have no common edge. Let  $P_1, \dots, P_m$  be paths in  $G$ . If  $P = P_1 + P_2 + \dots + P_m$ , and  $P_j$  are pairwise disjoint paths, then we call  $P_1 + \dots + P_m$  a *partition* of  $P$ .

**Definition 5.1.** Let  $P$  be a unicursal path of  $G$ . We say  $P = P_1 + P_2 + \dots + P_m$  is a *consistent partition*, if  $P_i$  is a path from  $a_i$  to  $a_{i+1}$  for each  $i \in \{1, 2, \dots, m\}$ .

Our first crucial assumption is:

**(A1)** *There exists a unicursal path  $P$  of  $G(\mathcal{S}, A, \beta)$  possessing a consistent partition  $P = P_1 + \dots + P_m$  of  $P$ .*

Denote the length of  $P_j$  by  $\ell_j$ . Then the path  $P_j$  ( $j = 1, \dots, m$ ) has the form

$$(5.3) \quad P_j = S_{j,1}(v_{j,1})S_{j,2}(v_{j,2}) \cdots S_{j,\ell_j}(v_{j,\ell_j}),$$

where  $S_{j,k} \in \{S_1, \dots, S_N\}$ , and  $v_{j,k} \in V$  for  $1 \leq k \leq \ell_j$ . Accordingly, the reverse path of  $P_j$  is defined to be (see Figure 15(right))

$$(5.4) \quad P_j^{-1} = S_{j,\ell_j}(v_{j,\ell_j}^{-1}) \cdots S_{j,2}(v_{j,2}^{-1})S_{j,1}(v_{j,1}^{-1}), \quad j = 1, \dots, m.$$

**5.2. Substitution rules.** By setting  $P_{v_j} = P_j$  in the edge-substitution method, we define a fine substitution rule  $\tau$  as

$$(5.5) \quad \tau : \begin{cases} v_j \mapsto P_j \\ v_j^{-1} \mapsto P_j^{-1}, \end{cases} \quad j = 1, \dots, m.$$

Orientations allow us to obtain more substitution rules  $\tau$ .

By identifying  $v_j$  and  $v_j^{-1}$  in  $\tau$ , and concentrating on the ‘types’ of elements of  $\tau(v)$ , we obtain the coarse substitution rule defined by

$$\tau^* : v_j \longrightarrow |v_{j,1}| |v_{j,2}| \dots |v_{j,\ell_j}|, \quad j = 1, 2, \dots, m,$$

where we set  $|v_j| = |v_j^{-1}| = v_j$ . Then  $\tau^*$  is a substitution over the alphabet  $\{v_1, \dots, v_m\}$ .

We say the partition  $P = P_1 + \dots + P_m$  is *primitive*, if the corresponding coarse substitution  $\tau^*$  is primitive. Our second assumption is:

**(A2)** *The partition  $P = P_1 + \dots + P_m$  is primitive.*

**Example 5.1.** Consider the Sierpiński gasket. Let  $\beta = (1, -1, -1)$ . Figure 15(left) illustrates a consistent partition  $P = P_1 + P_2 + P_3$ , where the sub-paths are colored by red, blue and green, respectively. Precisely,

$$(5.6) \quad \begin{aligned} P_1 &= S_1(v_1)S_2(v_3^{-1})S_2(v_2^{-1}), \\ P_2 &= S_2(v_1^{-1})S_1(v_2)S_3(v_3^{-1}), \\ P_3 &= S_3(v_2^{-1})S_3(v_1^{-1})S_1(v_3). \end{aligned}$$

The coarse substitution rule is

$$\tau^* : \begin{cases} v_1 \mapsto v_1 v_3 v_2, \\ v_2 \mapsto v_1 v_2 v_3, \\ v_3 \mapsto v_2 v_1 v_3, \end{cases}$$

and it is primitive. Figure 16 illustrates  $\tau^3(\Lambda_0)$ .

**5.3. GIFS induced by a consistent partition.** As a particular case of the edge-substitution method, the substitution rule (5.5) induces a linear GIFS:

$$(5.7) \quad \begin{cases} E_{v_j} \doteq S_{j,1}(E_{v_{j,1}}) \cup S_{j,2}(E_{v_{j,2}}) \cup \dots \cup S_{j,\ell_j}(E_{v_{j,\ell_j}}), \\ E_{v_j^{-1}} \doteq S_{j,\ell_j}(E_{v_{j,\ell_j}^{-1}}) \cup \dots \cup S_{j,2}(E_{v_{j,2}^{-1}}) \cup S_{j,1}(E_{v_{j,1}^{-1}}). \end{cases} \quad 1 \leq j \leq m,$$

Clearly, the state set is  $V = \{v_1, \dots, v_m\} \cup \{v_1^{-1}, \dots, v_m^{-1}\}$ . It is seen that, for a state  $v_j$ , there are  $\ell_j$  bridges emanating from it, and the  $k$ -th bridge is

$$(v_j, k, S_{j,k}, v_{j,k}),$$



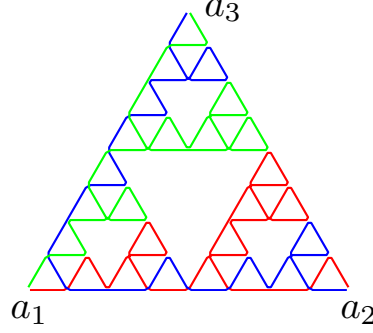


FIGURE 16. The third approximation of the Sierpiński gasket

whose components are initial state, order, associated contraction mapping and the terminate state of the bridge. In the same manner, the  $k$ -th bridge emanating from  $v_j^{-1}$  is denoted by

$$(v_j^{-1}, k, S_{j, \ell_j - k + 1}, v_{j, \ell_j - k + 1}^{-1}).$$

This defines the bridge set, and we denoted it by  $\mathcal{E}$ . We shall denote the GIFS (5.7) by

$$(5.8) \quad (V, \mathcal{E}, \mathcal{G}, \prec),$$

and call it the *GIFS induced by the partition*  $P = P_1 + \dots + P_m$ .

First, we give some basic properties of the induced GIFS.

**Lemma 5.1.** (i)  $E_{v_j} = E_{v_j^{-1}}$  for  $j = 1, 2, \dots, m$ .

(ii) For an arbitrary state  $v \in V$ , there are exactly  $N$  bridges ending at  $v$ , and the associated similitudes of them are exactly  $S_1, S_2, \dots, S_N$ .

(iii)  $K = \bigcup_{j=1}^m E_{v_j}$ .

*Proof.* (i) Set  $F_{v_j} = E_{v_j^{-1}}$  and  $F_{v_j^{-1}} = E_{v_j}$ . Clearly  $\{F_v\}_{v \in V}$  also satisfy the set equation (5.7). Hence (i) follows from the uniqueness of the invariant sets.

(ii) Since  $P$  is a unicursal path, for each  $v \in V$  and  $i \in \{1, \dots, m\}$ , the edge  $S_i(v)$  appears and appears only once in  $P$ , and it follows that the cylinder  $S_i(E_v)$  appears and appears only once in the right-hand side of (5.7).

(iii) Taking the union of both sides of (5.7) and using the conclusion of (ii), we have

$$(5.9) \quad \bigcup_{v \in V} E_v = \bigcup_{j=1}^m \bigcup_{k=1}^{\ell_j} S_{j,k}(E_{v_{j,k}} \cup E_{v_{j,k}^{-1}}) = \bigcup_{i=1}^N S_i(\bigcup_{v \in V} E_v).$$

Therefore, by the uniqueness of the invariant set of an IFS, we obtain

$$K = \bigcup_{v \in V} E_v = \bigcup_{j=1}^m E_{v_j},$$

where the last equality holds since  $E_{v_j} = E_{v_j^{-1}}$ .  $\square$

**5.4. The Open set condition.** Next, we prove that the induced GIFS  $(V, \mathcal{E}, \mathcal{G}, \prec)$  satisfies the OSC.

**Theorem 5.1.** *Let  $\mathcal{S}$  be an IFS fulfilling the OSC and possessing a finite skeleton  $A$ . Let  $P = P_1 + \cdots + P_m$  be a consistent and primitive partition of  $G(S, A, \beta)$ . Then GIFS (5.8) induced by the partition satisfies the OSC, and  $0 < \mathcal{H}^s(E_v) < \infty$  for all  $v \in V$ , where  $s = \dim_H K$ .*

*Proof.* First, we show that the similarity dimension of the induced GIFS is  $s = \dim_H K$ . Let  $c_j$  be the contraction ratio of  $S_j$ . Then  $s$  fulfills the equation  $\sum_{j=1}^N c_j^s = 1$  and  $0 < \mathcal{H}^s(K) < \infty$ , since  $\mathcal{S}$  satisfies the OSC.

The associate matrix  $M(s)$  of the induced GIFS  $(V, \mathcal{E}, \mathcal{F}, \prec)$  is

$$(5.10) \quad \left( \sum_{e \in \Gamma_{uv}} r_e^s \right)_{u, v \in V}.$$

By Lemma 5.1(ii), we have

$$\sum_{u \in V} \sum_{e \in \Gamma_{uv}} r_e^s = \sum_{i=1}^N c_i^s = 1, \quad v \in V,$$

namely, the sum of entries of each row of  $M(s)$  is 1. By Perron-Frobenius Theorem (see Lemma 5.3 below), the spectral radius of  $M(s)$  equals 1, that is,  $s$  is the similarity dimension of the induced GIFS.

We identify  $E_{v_j}$  and  $E_{v_j^{-1}}$  in the induced GIFS (5.7) and forget the order, then we obtain a *simplified* GIFS:

$$(5.11) \quad E_{v_j} = S_{j,1}(E_{|v_{j,1}|}) \cup S_{j,2}(E_{|v_{j,2}|}) \cup \cdots \cup S_{j,\ell_j}(E_{|v_{j,\ell_j}|}), \quad j = 1, \dots, m.$$

We have the following facts concerning the simplified GIFS:

(i) The base graph of the simplified GIFS is strongly connected, since the coarse substitution  $\tau^*$  is primitive.

(ii) There exists  $j \in \{1, \dots, m\}$  such that  $\mathcal{H}^s(E_{v_j}) > 0$ , since the union  $K = \bigcup_{j=1}^m E_{v_j}$  is an  $s$ -set.

(iii) The simplified GIFS has the same similarity dimension as the induced GIFS, which is  $s$ . This is true because the sum of each row of the associated matrix  $\bar{M}(s)$  of the simplified GIFS is still 1, where

$$\bar{M}(s) = \left( \sum_{e \in \Gamma_{uv}} r_e^s + \sum_{e \in \Gamma_{uv^{-1}}} r_e^s \right)_{u, v \in \{v_1, \dots, v_m\}}.$$

Item (i)-(iii) verify the conditions of Lemma 5.2 in below, hence the simplified GIFS satisfies the OSC, which implies  $0 < \mathcal{H}^s(E_{v_j}) < \infty$  for all  $j = 1, \dots, m$  (again by Lemma 5.2). Finally, we observe that if the simplified GIFS satisfies the OSC with open sets  $U_1, \dots, U_m$ , then clearly the induced GIFS satisfies the OSC with the open sets  $U_1, \dots, U_m, U_1, \dots, U_m$ .  $\square$

The following lemma is proved in [29] (the ‘only if’ part) and [25] (the ‘if’ part).

**Lemma 5.2.** *Let  $\mathcal{G}$  be a GIFS, with a strongly connected graph, invariant sets  $\{E_j\}_{j=1}^N$  and self-similar dimension  $\delta$ . Then  $\mathcal{G}$  satisfies the open set condition if and only if*

$$0 < \mathcal{H}^\delta(E_j) < +\infty$$

for some  $1 \leq j \leq N$  (or for all  $1 \leq j \leq N$ ).

The following is a part of the Perron-Frobenius Theorem, see for example [33].

**Lemma 5.3.** (Perron-Frobenius Theorem) *Let  $A = [a_{ij}]$  be a non-negative  $k \times k$  matrix.*

(i) *There is a non-negative eigenvalue  $\lambda$  such that it is the spectral radius  $A$ .*

(ii) *We have  $\min_i \left( \sum_{j=1}^k a_{ij} \right) \leq \lambda \leq \max_i \left( \sum_{j=1}^k a_{ij} \right)$ .*

**5.5. Pure cell.** In this subsection, we investigate when  $K = \bigcup_{j=1}^m E_{v_j}$  is a disjoint union in Hausdorff measure. To this end, we introduce the pure cells.

Let  $I \in \{1, \dots, N\}^n$  be a word of length  $n$ . We call the set of edges

$$\{S_I(v_j); j = 1, \dots, m\} \text{ or } \{S_I(v_j^{-1}); j = 1, \dots, m\}$$

an (positive or negative)  $S_I$ -cell.

**Definition 5.2.** An  $S_I$ -cell is called a *pure cell* with respect to a partition  $P = P_1 + \dots + P_m$ , if there exists  $u \in V$  such that all the edges in the  $S_I$ -cell belong to  $\tau^n(u)$ , where  $n = |I|$  and  $\tau$  is the fine substitution rule.

Our third assumption is:

**(A3).** *The partition  $P = P_1 + \dots + P_m$  possesses at least one pure cell.*

**Theorem 5.2.** *Suppose the conditions of Theorem 5.1 are fulfilled, and the above assumption (A3) holds. Then  $K = \bigcup_{j=1}^m E_{v_j}$  is a disjoint union in the  $s$ -dimensional Hausdorff measure, where  $s = \dim_H K$ .*

*Proof.* Suppose that the  $S_I$ -cell is a pure cell, *i.e.*, all the edges of the  $S_I$ -cell belong to  $\tau^n(u)$  for some  $u \in V$ . Without loss of generality, we may assume that the orientation of the  $S_I$ -cell is positive. Then all  $S_I(E_{v_j})$  ( $j = 1, \dots, m$ ) appear in the right hand side of the  $n$ -th iteration of the set equation (5.7) corresponding to  $E_u$ , hence  $\bigcup_{j=1}^m S_I(E_{v_j})$  is a disjoint union in the  $s$ -dimensional Hausdorff measure, since the induced GIFS satisfies the OSC by Theorem 5.1. It follows that  $\bigcup_{j=1}^m E_{v_j}$  is also a disjoint union in Hausdorff measure.  $\square$

**5.6. Proof of Theorem 1.3.** By Theorem 3.1, the induced GIFS is a linear GIFS. By Theorem 5.1, the induced GIFS satisfies the OSC, and all the invariant sets are  $s$ -sets.

Let  $h_j = \mathcal{H}^s(E_j)$ , and  $h = \sum_{j=1}^m h_j$ . By Theorem 1.1, for each  $j \in \{1, \dots, m\}$ , there exists an optimal parametrization  $\varphi_j : [0, h_j] \mapsto E_{v_j}$  such that  $\varphi_j(0) = a_j$ ,  $\varphi_j(h_j) = a_{j+1}$ .

Let  $\varphi : [0, h] \rightarrow K$  be the curve obtained by joining all the  $\varphi_i$  one by one. Then  $\varphi$  is an optimal parametrization of  $K$ , by Theorem 5.2.  $\square$

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